

# Dynamic Spectrum Management: A Complete Complexity Characterization

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## Abstract

Consider a multi-user multi-carrier communication system where multiple users share multiple discrete subcarriers. To achieve high spectrum efficiency, the users in the system must choose their transmit power dynamically in response to fast channel fluctuations. Assuming perfect channel state information, two formulations for the spectrum management (power control) problem are considered in this paper: the first is to minimize the total transmission power subject to all users' transmission data rate constraints, and the second is to maximize the min-rate utility subject to individual power constraints at each user. It is known in the literature that both formulations of the problem are polynomial time solvable when the number of subcarriers is one and strongly NP-hard when the number of subcarriers are greater than or equal to three. However, the complexity characterization of the problem when the number of subcarriers is two has been missing for a long time. This paper answers this long-standing open question: both formulations of the problem are strongly NP-hard when the number of subcarriers is two.

## Index Terms

Complexity theory, multi-carrier communication system, spectrum management, strong NP-hard.

## I. INTRODUCTION

In multi-carrier (or multi-tone) communication systems, the transmission frequency spectrum is partitioned into a number of orthogonal subcarriers on which parallel data can be simultaneously transmitted without causing interferences with each other. However, different users interfere

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with each other on the same subcarrier. The common examples of multi-carrier systems include wireless orthogonal frequency division multiplex (OFDM) systems (such as the 802.11) and wireline discrete multi-tone (DMT) systems (such as the digital subscriber line (DSL) system). In both of OFDM and DMT systems, a pair of discrete Fourier transform (DFT) and inverse discrete Fourier transform (IDFT) is used to effectively decompose the frequency-selective wideband channel into a group of non-selective narrowband subcarriers, which makes them robust against large delay spreads by preserving orthogonality in the frequency domain [1], [2].

Spectrum management, also called spectrum balancing or power control, is a central issue in the design of interference-limited multi-user multi-carrier communication systems. This is because in such systems the achievable data rate of each user depends not only on its own power allocation but also on the power allocation of all other users. The spectrum management problem in multi-user multi-carrier communication systems is often formulated as an optimization problem such as the system utility maximization problem subject to power budget constraints or the total power minimization problem subject to Quality-of-Service (QoS) constraints.

The spectrum management problem in the interference-limited multi-user multi-carrier communication system has been extensively studied; see [3]–[20] and references therein. The authors of [3] showed that the problem (under various optimization models) is (strongly) NP-hard when the number of subcarriers is greater than or equal to three. They also identified several subclasses of the problem which are polynomial time solvable when the number of subcarriers is one, such as the min-rate utility maximization problem and the total power minimization problem. However, the complexity characterization of the problem for the case where the number of subcarriers is two was missed for a long time in the literature.

The complexity results in [3] suggest that there are not polynomial time algorithms which can solve the general spectrum management problem to global optimality (unless  $P=NP$ ), and determining an approximately optimal or locally optimal spectrum management strategy in polynomial time is more realistic in practice (especially when a very fast responsiveness is required [15]). Therefore, various (heuristic) algorithms [3]–[20], including iterative water-filling algorithms, dual decomposition algorithms, and successive convex/concave approximation algorithms, have been proposed for solving the problem.

In this paper, we focus on the characterization of the computational complexity status of the spectrum management problem for the multi-user multi-carrier communication system. In par-

ticular, we consider two formulations of the problem. The first one is the problem of minimizing the total transmission power subject to all users' QoS constraints. The second one is the problem of maximizing the minimum rate among all users while respecting the total transmission power constraint of each user. The main contribution of this paper is to answer a long-standing open question: both aforementioned formulations of the spectrum management problem are strongly NP-hard when the number of subcarriers is two.

## II. PROBLEM FORMULATION

Consider a multi-user multi-carrier communication system, where there are  $K$  users (transmitter-receiver pairs) sharing  $N$  discrete subcarriers. Denote the set of users and the set of subcarriers by  $\mathcal{K} = \{1, 2, \dots, K\}$  and  $\mathcal{N} = \{1, 2, \dots, N\}$ , respectively. For any  $k \in \mathcal{K}$  and  $n \in \mathcal{N}$ , suppose  $s_k^n \in \mathbb{C}$  to be the symbol that transmitter  $k$  wishes to send to receiver  $k$  on subcarrier  $n$ , then the received signal  $\hat{s}_k^n$  at receiver  $k$  on subcarrier  $n$  can be expressed by

$$\hat{s}_k^n = \sum_{j \in \mathcal{K}} h_{k,j}^n s_j^n + z_k^n,$$

where  $h_{k,j}^n \in \mathbb{C}$  is the channel coefficient between the  $j$ -th transmitter and the  $k$ -th receiver on subcarrier  $n$  and  $z_k^n \in \mathbb{C}$  is the additive white Gaussian noise (AWGN) with distribution  $\mathcal{CN}(0, \eta_k^n)$ . Denoting the power of  $s_k^n$  by  $p_k^n$ ; i.e.,  $p_k^n := |s_k^n|^2$ , the received power at receiver  $k$  on subcarrier  $n$  is given by

$$\sum_{j \in \mathcal{K}} g_{k,j}^n p_j^n + \eta_k^n, \quad k \in \mathcal{K}, \quad n \in \mathcal{N},$$

where  $g_{k,j}^n := |h_{k,j}^n|^2$  stands for the channel gain between the  $j$ -th transmitter and the  $k$ -th receiver on subcarrier  $n$ . Treating interference as noise, we can write the SINR of receiver  $k$  on subcarrier  $n$  as

$$\text{SINR}_k^n = \frac{g_{k,k}^n p_k^n}{\sum_{j \neq k} g_{k,j}^n p_j^n + \eta_k^n}, \quad k \in \mathcal{K}, \quad n \in \mathcal{N},$$

and transmitter  $k$ 's achievable data rate  $R_k$  (nats/sec) as

$$R_k = \sum_{n \in \mathcal{N}} \ln \left( 1 + \frac{g_{k,k}^n p_k^n}{\sum_{j \neq k} g_{k,j}^n p_j^n + \eta_k^n} \right), \quad k \in \mathcal{K}. \quad (1)$$

In this paper, we consider the following two formulations of the spectrum management problem:

$$\begin{aligned}
& \min_{\{p_k^n\}} \sum_{k \in \mathcal{K}} \sum_{n \in \mathcal{N}} p_k^n \\
& \text{s.t.} \quad R_k \geq \gamma_k, \quad k \in \mathcal{K}, \\
& \quad \sum_{n \in \mathcal{N}} p_k^n \leq \bar{p}_k, \quad k \in \mathcal{K}, \\
& \quad p_k^n \geq 0, \quad k \in \mathcal{K}, \quad n \in \mathcal{N},
\end{aligned} \tag{2}$$

or

$$\begin{aligned}
& \max_{\{p_k^n\}} \min_{k \in \mathcal{K}} \{R_k\} \\
& \text{s.t.} \quad \sum_{n \in \mathcal{N}} p_k^n \leq \bar{p}_k, \quad k \in \mathcal{K}, \\
& \quad p_k^n \geq 0, \quad k \in \mathcal{K}, \quad n \in \mathcal{N},
\end{aligned} \tag{3}$$

where  $\gamma_k$  is the desired transmission rate target of user  $k$  and  $\bar{p}_k$  is the power budget of transmitter  $k$ . Formulation (2) minimizes the total transmission power of all users on all subcarriers and formulation (3) maximizes the minimum transmission rate among all users.

### III. COMPLEXITY ANALYSIS

In this section, we first briefly introduce complexity theory in Section III-A. Then, we review existing complexity results of problems (2) and (3) and show that both problems are strongly NP-hard when the number of subcarriers is two in Section III-B.

#### A. A Brief Introduction to Complexity Theory

In computational complexity theory [21]–[24], a problem is said to be NP-hard if it is at least as hard as any problem in the class NP (problems that are solvable in Nondeterministic Polynomial time). NP-complete problems are the hardest problems in NP in the sense that if any NP-complete problem is solvable in polynomial time, then each problem in NP is solvable in polynomial time. A problem is strongly NP-hard (strongly NP-complete) if it is NP-hard (NP-complete) and it can not be solved by a pseudo-polynomial time algorithm. An algorithm that solves a problem is called a *pseudo-polynomial* time algorithm if its time complexity function is bounded above by a polynomial function related to both of the length and the numerical values of the given data of the problem. This is in contrast to the polynomial time algorithm whose

time complexity function depends only on the length of the given data of the problem. It is widely believed that there can not exist a polynomial time algorithm to solve any NP-complete, NP-hard, or strongly NP-hard problem (unless  $P=NP$ ).

The standard way to prove an optimization problem is NP-hard is to establish the NP-hardness of its corresponding feasibility problem or decision problem. The latter is the problem to decide if the global minimum (maximum) of the optimization problem is below (above) a given threshold or not. To show a decision problem  $\mathcal{P}_2$  is NP-hard, we usually follow three steps: 1) choose a suitable NP-complete decision problem  $\mathcal{P}_1$ ; 2) construct a polynomial time transformation from any instance of  $\mathcal{P}_1$  to an instance of  $\mathcal{P}_2$ ; 3) prove under this transformation that any instance of problem  $\mathcal{P}_1$  is true if and only if the constructed instance of problem  $\mathcal{P}_2$  is true. See [21]–[24] for more on complexity theory.

#### B. Strong NP-Hardness of Problems (2) and (3) when $N = 2$

Both problems (2) and (3) are polynomial time solvable when  $N = 1$ . More specifically, when  $N = 1$ , problem (2) is equivalent to

$$\begin{aligned} \min_{\{p_k\}} \quad & \sum_{k \in \mathcal{K}} p_k \\ \text{s.t.} \quad & g_{k,k} p_k \geq (\exp(\gamma_k) - 1) \left( \sum_{j \neq k} g_{k,j} p_j + \eta_k \right), \quad k \in \mathcal{K}, \\ & \bar{p}_k \geq p_k \geq 0, \quad k \in \mathcal{K}, \end{aligned} \tag{4}$$

which is a linear program (solvable in polynomial time). When  $N = 1$ , problem (3) reduces to

$$\begin{aligned} \max_{\tau, \{p_k\}} \quad & \tau \\ \text{s.t.} \quad & g_{k,k} p_k \geq \tau \left( \sum_{j \neq k} g_{k,j} p_j + \eta_k \right), \quad k \in \mathcal{K}, \\ & \bar{p}_k \geq p_k \geq 0, \quad k \in \mathcal{K}, \end{aligned} \tag{5}$$

which is polynomial time solvable by using a binary search on  $\tau$ ; see [3, Theorem 2]. In fact, both problems (2) and (3) are also polynomial time solvable (by the water-filling algorithm) when  $K = 1$  (i.e., there is only a single user in the system) [5, Theorem 4.1].

Problems (2) and (3) become computationally intractable when the number of subcarriers is greater than or equal to three. In particular, it is shown in [3, Theorem 2] that problem (2) is strongly NP-hard when  $N \geq 3$ . By using the same argument as in the proof of [3, Theorem 2],

one can also show the strong NP-hardness of problem (3) with  $N \geq 3$ . However, the complexity characterization of problems (2) and (3) with  $N = 2$  has been missing for a long time in the literature. In this subsection, we answer this open question and show that both of problems (2) and (3) remain strongly NP-hard when  $N = 2$ .

The NP-hardness proof of problems (2) and (3) for the case  $N = 2$  is based on a polynomial time transformation from the MAX-2UNANIMITY problem, which was first introduced in [25]. To describe the problem, we first define the UNANIMITY property of a disjunctive clause. Recall that for a given set of Boolean variables, a literal is defined as either a Boolean variable or its negation, while a disjunctive clause refers to a logical expression consisting of the logical “OR” of literals.

*Definition 3.1 (UNANIMOUS):* For a given truth assignment to a set of Boolean variables, a disjunctive clause is said to be unanimous if all literals in the clause have the same value (whether it is the True or the False value).

*Definition 3.2 (MAX-2UNANIMITY):* Given a positive integer  $M$  and  $m$  disjunctive clauses defined over  $n$  Boolean variables, where the number of literals in each clause is 2, the MAX-2UNANIMITY problem is to check whether there exists a truth assignment such that the number of unanimous disjunctive clauses is at least  $M$ .

*Lemma 3.1 ([25]):* MAX-2UNANIMITY problem is NP-complete.

To show the NP-hardness of problems (2) and (3), we also need the following lemma, whose proof is relegated to Appendix A.

*Lemma 3.2:* The points  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (1, 0, 0, 1)^T$  and  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (0, 1, 1, 0)^T$  are the only feasible solutions of

$$\begin{cases} \ln \left( 1 + \frac{p_1^1}{1 + p_2^1} \right) + \ln \left( 1 + \frac{p_1^2}{1 + p_2^2} \right) \geq \ln 2, \\ \ln \left( 1 + \frac{p_2^1}{1 + p_1^1} \right) + \ln \left( 1 + \frac{p_2^2}{1 + p_1^2} \right) \geq \ln 2, \\ p_1^1 + p_1^2 \leq 1, \quad p_2^1 + p_2^2 \leq 1, \\ p_1^1 \geq 0, \quad p_1^2 \geq 0, \quad p_2^1 \geq 0, \quad p_2^2 \geq 0. \end{cases} \quad (6)$$

We are now ready to prove our main results.

*Theorem 3.1:* Problem (2) is strongly NP-hard when  $N \geq 2$ .

*Proof:* Given any instance of the MAX-2UNANIMITY problem with clauses  $c_1, c_2, \dots, c_m$

defined over Boolean variables  $x_1, x_2, \dots, x_n$  and an integer  $M$ , we construct below a multi-user multi-carrier interference channel with  $2n+m$  users and 2 subcarriers, where the Boolean variable  $x_i$  ( $i = 1, 2, \dots, n$ ) corresponds to a pair of users, including user  $i$  (called “variable user”) and user  $n+i$  (called “auxiliary variable user”); each clause  $c_j$  ( $j = 1, 2, \dots, m$ ) corresponds to user  $2n+j$  (called “clause user”). Hence,  $\mathcal{K} = \{1, 2, \dots, 2n+m\}$  and  $\mathcal{N} = \{1, 2\}$ .

Now, we construct transmission rate expressions for all users. For  $i = 1, 2, \dots, n$ , let

$$R_i = \ln \left( 1 + \frac{p_i^1}{1 + p_{n+i}^1} \right) + \ln \left( 1 + \frac{p_i^2}{1 + p_{n+i}^2} \right) \quad (7)$$

and

$$R_{n+i} = \ln \left( 1 + \frac{p_{n+i}^1}{1 + p_i^1} \right) + \ln \left( 1 + \frac{p_{n+i}^2}{1 + p_i^2} \right); \quad (8)$$

for  $j = 1, 2, \dots, m$ , let

$$R_{2n+j} = \begin{cases} \ln \left( 1 + \frac{p_{2n+j}^1}{1 + p_{i_1}^1 + p_{i_2}^1} \right) + \ln \left( 1 + \frac{p_{2n+j}^2}{1 + p_{i_1}^2 + p_{i_2}^2} \right), & \text{if } c_j = x_{i_1} \vee x_{i_2}; \\ \ln \left( 1 + \frac{p_{2n+j}^1}{1 + p_{i_1}^1 + p_{n+i_2}^1} \right) + \ln \left( 1 + \frac{p_{2n+j}^2}{1 + p_{i_1}^2 + p_{n+i_2}^2} \right), & \text{if } c_j = x_{i_1} \vee \bar{x}_{i_2}; \\ \ln \left( 1 + \frac{p_{2n+j}^1}{1 + p_{n+i_1}^1 + p_{i_2}^1} \right) + \ln \left( 1 + \frac{p_{2n+j}^2}{1 + p_{n+i_1}^2 + p_{i_2}^2} \right), & \text{if } c_j = \bar{x}_{i_1} \vee x_{i_2}; \\ \ln \left( 1 + \frac{p_{2n+j}^1}{1 + p_{n+i_1}^1 + p_{n+i_2}^1} \right) + \ln \left( 1 + \frac{p_{2n+j}^2}{1 + p_{n+i_1}^2 + p_{n+i_2}^2} \right), & \text{if } c_j = \bar{x}_{i_1} \vee \bar{x}_{i_2}. \end{cases} \quad (9)$$

In the above, each user  $k$  is associated with two variables  $p_k^1$  and  $p_k^2$  for  $k \in \mathcal{K}$ . The noise power of all users on all subcarriers are 1. For each “variable user”  $i$ , it suffers interference from “auxiliary variable user”  $n+i$  on both subcarriers 1 and 2; each “auxiliary variable user”  $n+i$  suffers interference from “variable user”  $i$  on both subcarriers 1 and 2; each “clause user”  $2n+j$  suffers interference from “variable user”  $i_1$  and  $i_2$  and/or “auxiliary variable user”  $n+i_1$  and  $n+i_2$ , where  $c_j$  contains literals of  $x_{i_1}$  and  $x_{i_2}$ . To make the construction of transmission rate expressions clear, an illustrative example is given in Appendix B.

Moreover, let  $\gamma_k = \ln 2$  and  $\bar{p}_k = 1$  for all  $k \in \mathcal{K}$ . Then, the constructed instance of problem

(2) is

$$\begin{aligned}
& \min_{\{p_k^n\}} \sum_{k \in \mathcal{K}} \sum_{n \in \mathcal{N}} p_k^n \\
& \text{s.t.} \quad R_k \geq \ln 2, \quad k \in \mathcal{K}, \\
& \quad p_k^1 + p_k^2 \leq 1, \quad k \in \mathcal{K}, \\
& \quad p_k^n \geq 0, \quad k \in \mathcal{K}, \quad n \in \mathcal{N},
\end{aligned} \tag{10}$$

where  $R_k$  are given in (7), (8), and (9). Next, we show that there exists a truth assignment such that at least  $M$  clauses are satisfied unanimously for the given MAX-2UNANIMITY instance if and only if the optimal value of problem (10) is less than or equal to  $2n + M + 4(\sqrt{2} - 1)(m - M)$ .

If there exists a truth assignment such that  $M$  clauses in the MAX-2UNANIMITY problem are unanimous, we claim that the optimal value of problem (10) is less than or equal to  $2n + M + 4(\sqrt{2} - 1)(m - M)$ . Let  $\{x_i\}$  be the truth assignment such that  $M$  clauses are unanimous in the MAX-2UNANIMITY problem. We set

$$p_i^1 = p_{n+i}^2 = 1 - x_i, \quad p_i^2 = p_{n+i}^1 = x_i, \quad i = 1, 2, \dots, n.$$

With this, we can simply check that  $R_k \geq \ln 2$  for all  $k = 1, 2, \dots, 2n$ . Furthermore, we consider transmission rate requirements of the “clause variable”  $2n + j$  with  $j = 1, 2, \dots, m$ .

- If the clause  $c_j$  is unanimous, then we have either

$$R_{2n+j} = \ln(1 + p_{2n+j}^1) + \ln\left(1 + \frac{p_{2n+j}^2}{3}\right)$$

or

$$R_{2n+j} = \ln\left(1 + \frac{p_{2n+j}^1}{3}\right) + \ln(1 + p_{2n+j}^2).$$

In either cases, we can use a total transmission power of 1 to make  $R_{2n+j} \geq \ln 2$  satisfied (by setting  $(p_{2n+j}^1, p_{2n+j}^2)^T = (1, 0)^T$  in the former case and  $(p_{2n+j}^1, p_{2n+j}^2)^T = (0, 1)^T$  in the latter case).

- If the clause  $c_j$  is not unanimous, then we must have

$$R_{2n+j} = \ln\left(1 + \frac{p_{2n+j}^1}{2}\right) + \ln\left(1 + \frac{p_{2n+j}^2}{2}\right).$$

In this case, we can use a total transmission power of  $4(\sqrt{2} - 1)$  to make  $R_{2n+j} \geq \ln 2$  satisfied (by setting  $p_{2n+j}^1 = p_{2n+j}^2 = 2(\sqrt{2} - 1)$ ).



As a result, if there exists a truth assignment such that at least  $M$  clauses are satisfied unanimously, then the optimal value of problem (10) is less than or equal to  $2n + M + 4(\sqrt{2} - 1)(m - M)$ .

For the converse part, assuming that the optimal value of problem (10) is less than or equal to  $2n + M + 4(\sqrt{2} - 1)(m - M)$ , we claim that at least  $M$  clauses can be made unanimous. It follows from Lemma 3.2 that, for  $i = 1, 2, \dots, n$ , the optimal solution of problem (10) must be

$$(p_i^1, p_i^2, p_{n+i}^1, p_{n+i}^2)^T = (1, 0, 0, 1)$$

or

$$(p_i^1, p_i^2, p_{n+i}^1, p_{n+i}^2)^T = (0, 1, 1, 0).$$

This, together with (9), implies that the received total interferences at user  $2n + j$  must be exactly 2 for all  $j = 1, 2, \dots, m$ . More specifically, there might be two cases:

- Case 1: the received interference at user  $2n + j$  is equal 2 on one subcarrier and is equal to 0 on the other one;
- Case 2: the received interference at user  $2n + j$  on both subcarriers is equal to 1.

If Case 1 happens for user  $2n + j$ , then the required total transmission power satisfying  $R_{2n+j} \geq \ln 2$  is at least 1; while if Case 2 happens for user  $2n + j$ , then the required total transmission power satisfying  $R_{2n+j} \geq \ln 2$  is at least  $4(\sqrt{2} - 1)$ . By the assumption that the optimal value of problem (10) is less than or equal to  $2n + M + 4(\sqrt{2} - 1)(m - M)$ , we know that Case 1 must happen at least  $M$  times (Case 2 cannot happen more than  $m - M$  times). Moreover, it can be checked that

$$x_i = 1 - p_i^1, \quad i = 1, 2, \dots, n$$

is a truth assignment which makes at least  $M$  clauses in the MAX-2UNANIMITY problem satisfied unanimously.

Finally, the transformation from the MAX-2UNANIMITY problem to problem (10) can be finished in polynomial time. Since the MAX-2UNANIMITY problem is NP-complete (cf. Lemma 3.1), we conclude that the problem of checking the optimal value of problem (10) is less than or equal to  $2n + M + 4(\sqrt{2} - 1)(m - M)$  is strongly NP-hard. Hence, problem (2) is strongly NP-hard. ■

*Theorem 3.2:* Problem (3) is strongly NP-hard when  $N \geq 2$ .

*Proof:* The proof of Theorem 3.2 is similar to the one of Theorem 3.1. For succinctness, we just give the proof outline. Given any instance of the MAX-2UNANIMITY problem with clauses  $c_1, c_2, \dots, c_m$  defined over Boolean variables  $x_1, x_2, \dots, x_n$  and an integer  $M$ , we construct below a multi-user multi-carrier interference channel with  $2n + 2m + 1$  users and 2 subcarriers. In addition to “variable user”  $i$  and “auxiliary variable user”  $n + i$  for  $i = 1, 2, \dots, n$  and “clause user”  $2n + j$  for  $j = 1, 2, \dots, m$ , we also construct “auxiliary clause user”  $2n + m + j$  for  $j = 1, 2, \dots, m$  and “super user”  $2n + 2m + 1$ . Hence,  $\mathcal{K} = \{1, 2, \dots, 2n + 2m + 1\}$  and  $\mathcal{N} = \{1, 2\}$ .

We construct transmission rate expressions for all users as follows. For  $i = 1, 2, \dots, n$ , let  $R_i$  and  $R_{n+i}$  be the same as the ones in (7) and (8), respectively; for  $j = 1, 2, \dots, m$ , let  $R_{2n+j}$  be the same as the one in (9) and

$$R_{2n+m+j} = \begin{cases} \ln \left( 1 + \frac{p_{2n+m+j}^1}{1 + p_{n+i_1}^1 + p_{n+i_2}^1} \right) + \ln \left( 1 + \frac{p_{2n+m+j}^2}{1 + p_{n+i_1}^2 + p_{n+i_2}^2} \right), & \text{if } c_j = x_{i_1} \vee x_{i_2}; \\ \ln \left( 1 + \frac{p_{2n+m+j}^1}{1 + p_{n+i_1}^1 + p_{i_2}^1} \right) + \ln \left( 1 + \frac{p_{2n+m+j}^2}{1 + p_{n+i_1}^2 + p_{i_2}^2} \right), & \text{if } c_j = x_{i_1} \vee \bar{x}_{i_2}; \\ \ln \left( 1 + \frac{p_{2n+m+j}^1}{1 + p_{i_1}^1 + p_{n+i_2}^1} \right) + \ln \left( 1 + \frac{p_{2n+m+j}^2}{1 + p_{i_1}^2 + p_{n+i_2}^2} \right), & \text{if } c_j = \bar{x}_{i_1} \vee x_{i_2}; \\ \ln \left( 1 + \frac{p_{2n+m+j}^1}{1 + p_{i_1}^1 + p_{i_2}^1} \right) + \ln \left( 1 + \frac{p_{2n+m+j}^2}{1 + p_{i_1}^2 + p_{i_2}^2} \right), & \text{if } c_j = \bar{x}_{i_1} \vee \bar{x}_{i_2}; \end{cases} \quad (11)$$

and let

$$R_{2n+2m+1} = \ln \left( 1 + \frac{p_{2n+2m+1}^1}{1 + \sum_{j=1}^{2m} p_{2n+j}^1} \right) + \ln \left( 1 + \frac{p_{2n+2m+1}^2}{1 + \sum_{j=1}^{2m} p_{2n+j}^2} \right). \quad (12)$$

Moreover, let

$$\bar{p}_k = \begin{cases} 1, & \text{if } 1 \leq k \leq 2n; \\ 4(\sqrt{2} - 1), & \text{if } 2n + 1 \leq k \leq 2n + 2m; \\ 2(\sqrt{2} - 1) \left( 1 + 2n + M + 4(\sqrt{2} - 1)(m - M) \right), & \text{if } k = 2n + 2m + 1. \end{cases} \quad (13)$$

Then, the constructed instance of problem (3) is

$$\begin{aligned}
& \max_{\{p_k^n\}} \min_{k \in \mathcal{K}} \{R_k\} \\
& \text{s.t.} \quad p_k^1 + p_k^2 \leq 1, \quad k = 1, 2, \dots, 2n, \\
& \quad p_k^1 + p_k^2 \leq 4 \left( \sqrt{2} - 1 \right), \quad k = 2n+1, 2n+2, \dots, 2n+2m, \\
& \quad p_{2n+2m+1}^1 + p_{2n+2m+1}^2 \leq 2 \left( \sqrt{2} - 1 \right) \left( 1 + 2n + M + 4(\sqrt{2} - 1)(m - M) \right), \\
& \quad p_k^1 \geq 0, \quad p_k^2 \geq 0, \quad k \in \mathcal{K},
\end{aligned} \tag{14}$$

where  $R_k$  are given in (7), (8), (9), (11), and (12). Next, we show that there exists a truth assignment such that at least  $M$  clauses are satisfied unanimously for the given MAX-2UNANIMITY instance if and only if the optimal value of problem (14) is greater than or equal to 1.

For any  $j = 1, 2, \dots, m$ , since  $R_{2n+j}$  in (9) and  $R_{2n+m+j}$  in (11) are symmetric, it follows that  $p_{2n+j}^1 = p_{2n+m+j}^2$  and  $p_{2n+m+j}^1 = p_{2n+j}^2$  at the optimal solutions of problem (14). This shows that

$$\sum_{j=1}^{2m} p_{2n+j}^1 = \sum_{j=1}^{2m} p_{2n+j}^2 = \sum_{j=1}^m p_{2n+j}^1 + \sum_{j=1}^m p_{2n+j}^2$$

holds at the optimal solutions of problem (14), which, together with the fact

$$\bar{p}_{2n+2m+1} = 2 \left( \sqrt{2} - 1 \right) \left( 1 + 2n + M + 4(\sqrt{2} - 1)(m - M) \right),$$

implies that the optimal value of problem (14) is greater than or equal to 1 if and only if

$$\sum_{j=1}^m p_{2n+j}^1 + \sum_{j=1}^m p_{2n+j}^2$$

at the optimal solution is less than or equal to  $2n + M + 4(\sqrt{2} - 1)(m - M)$ . Since the problem of checking the latter is strongly NP-hard (cf. Theorem 3.1), we conclude that checking the optimal value of problem (14) is greater than or equal to 1 is strongly NP-hard. Therefore, problem (3) is strongly NP-hard. ■

Table I summarizes the complexity status of spectrum management problems (2) and (3).

#### IV. CONCLUSIONS

Dynamic spectrum management in accordance with fast channel fluctuations can significantly improve spectral efficiency of the multi-user multi-carrier communication system. A major challenge associated with spectrum management is to find, for a given channel state, the globally optimal spectrum management strategy to minimize the total transmission power or maximize the

TABLE I  
COMPLEXITY STATUS OF SPECTRUM MANAGEMENT FOR MULTI-USER MULTI-SUBCARRIER COMMUNICATION SYSTEMS

<div style="text-align: center;"># of Subcarriers \ Problem</div>	Total Power Minimization Problem (2)	Min-Rate Maximization Problem (3)
$N = 1$	Polynomial Time Solvable [3]	Polynomial Time Solvable [3]
$N \geq 2$	Strongly NP-hard (Theorem 3.1)	Strongly NP-hard (Theorem 3.2)

system utility. This paper has provided a complete complexity characterization of the spectrum management problem in the multi-user multi-subcarrier communication system. We have shown that both the total power minimization problem and the min-rate maximization problem are strongly NP-hard when the number of subcarriers is two, and thus answered a long-standing open question in the literature. The complexity results suggest that there are not polynomial time algorithms which can solve the general spectrum management problem to global optimality (unless P=NP) and it is more realistic to design efficient algorithms for finding an approximately optimal or locally optimal spectrum management strategy in polynomial time in practice.

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#### APPENDIX A

##### PROOF OF LEMMA 3.2

We first prove that  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (1, 0, 0, 1)^T$  and  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (0, 1, 1, 0)^T$  are the only two optimal solutions of problem

$$\begin{aligned}
 & \min_{p_1^1, p_1^2, p_2^1, p_2^2} \quad p_1^1 + p_1^2 + p_2^1 + p_2^2 \\
 & \text{s.t.} \quad \ln \left( 1 + \frac{p_1^1}{1 + p_2^1} \right) + \ln \left( 1 + \frac{p_1^2}{1 + p_2^2} \right) \geq \ln 2, \\
 & \quad \ln \left( 1 + \frac{p_2^1}{1 + p_1^1} \right) + \ln \left( 1 + \frac{p_2^2}{1 + p_1^2} \right) \geq \ln 2, \\
 & \quad p_1^1 \geq 0, p_1^2 \geq 0, p_2^1 \geq 0, p_2^2 \geq 0.
 \end{aligned} \tag{15}$$

The two rate constraints in problem (15) can be equivalently rewritten as

$$(1 + p_1^1 + p_2^1) (1 + p_1^2 + p_2^2) \geq 2 (1 + p_1^1) (1 + p_2^2)$$

and

$$(1 + p_1^1 + p_2^1) (1 + p_1^2 + p_2^2) \geq 2 (1 + p_1^1) (1 + p_2^1) .$$

Adding the above two inequalities together yields  $p_1^1 p_2^2 + p_1^2 p_2^1 \geq 1$ , which implies that problem

$$\begin{aligned} \min_{p_1^1, p_1^2, p_2^1, p_2^2} \quad & p_1^1 + p_1^2 + p_2^1 + p_2^2 \\ \text{s.t.} \quad & p_1^1 p_2^2 + p_1^2 p_2^1 \geq 1, \\ & p_1^1 \geq 0, p_1^2 \geq 0, p_2^1 \geq 0, p_2^2 \geq 0, \end{aligned} \tag{16}$$

is a relaxation of problem (15). If we can show  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (1, 0, 0, 1)^T$  and  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (0, 1, 1, 0)^T$  are the only two optimal solutions of problem (16), then they must be the only two optimal solutions of problem (15). This further implies that  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (1, 0, 0, 1)^T$  and  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (0, 1, 1, 0)^T$  are the only feasible points of

$$\begin{cases} \ln \left( 1 + \frac{p_1^1}{1 + p_2^1} \right) + \ln \left( 1 + \frac{p_1^2}{1 + p_2^2} \right) \geq \ln 2, \\ \ln \left( 1 + \frac{p_2^1}{1 + p_1^1} \right) + \ln \left( 1 + \frac{p_2^2}{1 + p_1^2} \right) \geq \ln 2, \\ p_1^1 + p_1^2 + p_2^1 + p_2^2 \leq 2, \\ p_1^1 \geq 0, p_1^2 \geq 0, p_2^1 \geq 0, p_2^2 \geq 0. \end{cases} \tag{17}$$

Since

$$\{(p_1^1, p_1^2, p_2^1, p_2^2) \geq 0 \mid p_1^1 + p_1^2 \leq 1, p_2^1 + p_2^2 \leq 1\} \subseteq \{(p_1^1, p_1^2, p_2^1, p_2^2) \geq 0 \mid p_1^1 + p_1^2 + p_2^1 + p_2^2 \leq 2\},$$

it follows that  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (1, 0, 0, 1)^T$  and  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (0, 1, 1, 0)^T$  are the only feasible points of (6).

It remains to prove that  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (1, 0, 0, 1)^T$  and  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (0, 1, 1, 0)^T$  are the only two optimal solutions of problem (16). It can be verified that the optimal solution of the following problem

$$\begin{aligned} \min_{p_1^1, p_1^2, p_2^1, p_2^2} \quad & 2 \left( \sqrt{p_1^1 p_2^2} + \sqrt{p_1^2 p_2^1} \right) \\ \text{s.t.} \quad & p_1^1 p_2^2 + p_1^2 p_2^1 \geq 1, \\ & p_1^1 \geq 0, p_1^2 \geq 0, p_2^1 \geq 0, p_2^2 \geq 0, \end{aligned}$$

must satisfy

$$p_1^1 p_2^2 = 1, \quad p_1^2 p_2^1 = 0$$

or

$$p_1^1 p_2^2 = 0, \quad p_1^2 p_2^1 = 1,$$

and its optimal value is 2. Since  $p_1^1 + p_2^2 \geq 2\sqrt{p_1^1 p_2^2}$  and  $p_1^2 + p_2^1 \geq 2\sqrt{p_1^2 p_2^1}$  and the above two inequalities hold true with “=” if and only if  $p_1^1 = p_2^2$  and  $p_1^2 = p_2^1$ , we conclude that the only points that achieve the objective value of problem (16) of being 2 are  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (1, 0, 0, 1)^T$  and  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (0, 1, 1, 0)^T$ . Hence,  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (1, 0, 0, 1)^T$  and  $(p_1^1, p_1^2, p_2^1, p_2^2)^T = (0, 1, 1, 0)^T$  are the only two optimal solutions of problem (16). This completes the proof of Lemma 3.2.

## APPENDIX B

### AN ILLUSTRATIVE EXAMPLE

In this example, there are four disjunctive clauses  $c_1 = x_1 \vee \bar{x}_2$ ,  $c_2 = x_1 \vee x_3$ ,  $c_3 = \bar{x}_2 \vee \bar{x}_4$ , and  $c_4 = \bar{x}_3 \vee x_4$  defined over four Boolean variables  $x_1, x_2, x_3$ , and  $x_4$ . Then there are 12 users in the constructed 2-carrier communication system, including 4 “variable user” (denoted as user 1, 2, 3, 4), 4 “auxiliary variable user” (denoted as user 5, 6, 7, 8), and 4 “clause user” (denoted as user 9, 10, 11, 12). In this case,  $\mathcal{K} = \{1, 2, \dots, 12\}$ , and all users’ rate expressions are given

as follows:

$$\begin{aligned}
R_1 &= \ln \left( 1 + \frac{p_1^1}{1 + p_5^1} \right) + \ln \left( 1 + \frac{p_1^2}{1 + p_5^2} \right), \\
R_2 &= \ln \left( 1 + \frac{p_2^1}{1 + p_6^1} \right) + \ln \left( 1 + \frac{p_2^2}{1 + p_6^2} \right), \\
R_3 &= \ln \left( 1 + \frac{p_3^1}{1 + p_7^1} \right) + \ln \left( 1 + \frac{p_3^2}{1 + p_7^2} \right), \\
R_4 &= \ln \left( 1 + \frac{p_4^1}{1 + p_8^1} \right) + \ln \left( 1 + \frac{p_4^2}{1 + p_8^2} \right), \\
R_5 &= \ln \left( 1 + \frac{p_5^1}{1 + p_1^1} \right) + \ln \left( 1 + \frac{p_5^2}{1 + p_1^2} \right), \\
R_6 &= \ln \left( 1 + \frac{p_6^1}{1 + p_2^1} \right) + \ln \left( 1 + \frac{p_6^2}{1 + p_2^2} \right), \\
R_7 &= \ln \left( 1 + \frac{p_7^1}{1 + p_3^1} \right) + \ln \left( 1 + \frac{p_7^2}{1 + p_3^2} \right), \\
R_8 &= \ln \left( 1 + \frac{p_8^1}{1 + p_4^1} \right) + \ln \left( 1 + \frac{p_8^2}{1 + p_4^2} \right), \\
R_9 &= \ln \left( 1 + \frac{p_9^1}{1 + p_1^1 + p_6^1} \right) + \ln \left( 1 + \frac{p_9^2}{1 + p_1^2 + p_6^2} \right), \\
R_{10} &= \ln \left( 1 + \frac{p_{10}^1}{1 + p_1^1 + p_3^1} \right) + \ln \left( 1 + \frac{p_{10}^2}{1 + p_1^2 + p_3^2} \right), \\
R_{11} &= \ln \left( 1 + \frac{p_{11}^1}{1 + p_6^1 + p_8^1} \right) + \ln \left( 1 + \frac{p_{11}^2}{1 + p_6^2 + p_8^2} \right), \\
R_{12} &= \ln \left( 1 + \frac{p_{12}^1}{1 + p_7^1 + p_4^1} \right) + \ln \left( 1 + \frac{p_{12}^2}{1 + p_7^2 + p_4^2} \right).
\end{aligned}$$

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